

A Topological String: The Rasetti-Regge Lagrangian, Topological Quantum Field Theory, and Vortices in Quantum Fluids

A. D. Speliotopoulos*

*National Research Council, Board on Physics and Astronomy,
2101 Constitution Avenue, NW, Washington, DC 20418[†]*

(Dated: July 29, 2002)

Abstract

The kinetic part of the Rasetti-Regge action I_{RR} for vortex lines is studied and links to string theory are made. It is shown that both I_{RR} and the Polyakov string action I_{Pol} can be constructed with the same field X^μ . Unlike I_{NG} , however, I_{RR} describes a Schwarz-type topological quantum field theory. Using generators of classical Lie algebras, I_{RR} is generalized to higher dimensions. In all dimensions, the momentum 1-form \mathbf{P} constructed from the canonical momentum for the vortex belongs to the first cohomology class $H^1(M, \mathbb{R}^m)$ of the worldsheet M swept-out by the vortex line. The dynamics of the vortex line thus depend directly on the topology of M . For a vortex ring, the equations of motion reduce to the Serret-Frenet equations in \mathbb{R}^3 , and in higher dimensions they reduce to the Maurer-Cartan equations for $so(m)$.

INTRODUCTION

That vortex lines play an important role in many physical systems is well-known [1, 2]. Following the original ideas of Onsager [3] and Feynman [4], researchers have even used vortex rings in an attempt to explain the underlying cause of the the lambda transition for superfluid He^4 [5, 6]. More recently, the discovery of Bose-Einstein condensates (BEC) [7, 8] has renewed interest in the study of vortex lines, and within the last year vortex excitations have been seen experimentally in BEC [9, 10, 11, 12]. There has been a corresponding theoretical interest in the formation and stability of vortex lines in BEC (see, for example, [13] and [14]).

While experimental studies of vortex lines in quantum fluids have been remarkable, theoretical understanding of the dynamics and interactions of vortex lines on a quantum level has proceeded at a much slower pace. Recent theoretical work on vortices in BEC has mostly been focused only on the *formation* and *stability* of vortices in the condensate and not on the properties and dynamics of the vortices once they are formed.

Much of the efforts in developing a deeper understanding of the dynamics of vortex lines on a quantum level have been based on the work of M. Rasetti and T. Regge. Using arguments from classical fluid dynamics of ideal fluids, they [15] proposed a lagrangian for studying the quantum theory of vortex lines in quantum fluids in three dimensions. Vortex lines are treated as extended objects and like a string in string theory, a vortex line sweeps out a two-dimensional worldsheet M as it propagates in time. Current algebra methods leading to the study of $Sdiff(\mathbb{R}^3)$, the diffeomorphic group in \mathbb{R}^3 , were then used by them and subsequent researchers [16, 17, 18, 19, 20, 21, 22, 23] in an effort to quantize the field.

In this paper we propose a different approach to understanding the dynamics of vortex lines. Focusing on a single vortex, we start with the field X^μ and make use of the $so(3)$ Lie algebra to rewrite the kinetic part of the full Rasetti-Regge action I_{RR} in terms of differential forms, and demonstrate how I_{RR} is related to the Polyakov form [24] of the Nambu-Goto action I_{Pol} [25, 26] (see also [27] for a different approach). It is then straightforward to see that while I_{Pol} defines a propagating string, I_{RR} defines, when quantized, a Schwarz-type topological quantum field theory (TQFT) [28, 29, 30]. Indeed, I_{RR} is very similar in form to the Chern-Simons lagrangian. Making use of other classical Lie algebras, we then extend this construction to higher dimensions; the linkage between I_{Pol} and I_{RR} still hold (see also

[31]). However, unlike I_{Pol} , which can be constructed in any dimension, I_{RR} exists only in a discrete number of dimension corresponding to the dimensions of the Lie algebra used in its construction.

Using this approach, it becomes clear that the understanding of the quantum-and thus statistical behavior-of vortex lines will be the first real world application of TQFT. Conversely, the vortex system provides a means of studying experimentally a TQFT for the first time. The purpose of this paper is thus to make the connection between I_{RR} and TQFT, and our approach is strictly classical, and our analysis formal. Nonetheless, a great deal can immediately be discerned about the properties of vortex lines from the fact that it is a TQFT and this classical analysis.

As is well known, a TQFT does not define a dynamical system in a traditional sense; a single vortex line does not, strictly speaking, have dynamical variables that evolve with time. TQFT's are interesting nonetheless, [28, 29]. While our approach is strictly classical, even at this level we find deep connections between topology and the dynamics of vortices. Indeed, we show that the momentum 1-form \mathbf{P} constructed from the canonical momentum of the vortex line belongs to the first cohomology class of M ; the dynamics of vortices depend directly on the topology of M . Going further, we show formally that the solution to the equations of motion in three-dimensions reduces to the Serret-Frenet equations for arbitrary curves in \mathbb{R}^3 . These equations are themselves equivalent to the equation of motion of a charged particle constrained to move on a unit sphere in the the presence of a dyon located at the center. In higher dimensions the equations of motion reduce to the Maurer-Cartan equations for $so(m)$. The Maurer-Cartan 1-forms can be interpreted as a “pure gauge” non-abelian vector potential, and as is the case for TQFT, we are working with flat vector bundles. With this analogy, explicit solutions of the equations of motion can be found using Wilson path ordering.

Connections between I_{RR} and string theory goes beyond the construction of I_{Pol} , however. A term of the form $\int B_{\mu\nu} dX^\mu \wedge dX^\nu$, where $B_{\mu\nu}$ is a antisymmetric tensor functional of the string field, was added to I_{pol} by Callan, Friedan, Martinec, and Perry [32] in their background field treatment of string. $B_{\mu\nu}$ generates an all-pervasive magnetic field in space-time. While similar in form to I_{RR} , in their treatment the specific functional dependence of $B_{\mu\nu}$ on X^μ was determined by requiring that the trace anomaly of the *total* string action vanish. This resulted in a $B_{\mu\nu}$ that is dramatically different from what is considered here.

Along similar lines, Giveon, Rabinovici, and Veneziano [33] also considered a $B_{\mu\nu}$ term in the string lagrangian, but relaxed the trace anomaly condition and considered the effect of *constant* $B_{\mu\nu}$ on the string.

GENERAL CONSTRUCTION OF I_{RR}

We begin with a classical, real Lie algebra \mathfrak{g} with generators \mathbf{T}_a such that $[\mathbf{T}_a, \mathbf{T}_b] = f_{ab}{}^c \mathbf{T}_c$, where $f_{ab}{}^c$ are the structure constants for \mathfrak{g} , and indices run from 1 to m , the dimension of the \mathfrak{g} . \mathbf{T}^a is represented by matrices and following the convention in [34], the Killing form $h_{ab} \equiv \text{Tr} \{ \mathbf{T}_a \mathbf{T}_b \} = -\delta_{ab}$ is used to raise and lower indices: $A_a = h_{ab} A^b = -A^b$. With this orthonormality condition, we can use the set $\{ \mathbf{T}_a \}$ as a natural basis for \mathbb{R}^m , with $\mathbf{V} \in \mathbb{R}^m$ given by $\mathbf{V} = V^a \mathbf{T}_a$. The inner product on \mathbb{R}^m is then $\langle \mathbf{V}, \mathbf{U} \rangle \equiv -\text{Tr} \{ \mathbf{V} \mathbf{U} \}$ for $\mathbf{V}, \mathbf{U} \in \mathbb{R}^m$. Furthermore, using the identity matrix \mathbf{I} of \mathfrak{g} we can extend this construction to the $m+1$ -dimensional Minkowski space Min by taking $V = V_0 \mathbf{I} / \sqrt{m} + \mathbf{V}$ for $V \in Min$. When $\mathfrak{g} = su(2)$, this is just a representation of Minkowski space by the quaternions.

We next consider an extended object $X^\mu(x^0, x^1)$ sweeping-out a 2-d surface M in Min where x^0 and x^1 are the spatial and time coordinates on M . Taking $X = X_0 \mathbf{I} / \sqrt{m} + X_a \mathbf{T}^a$, the usual Nambu-Goto string action is obtained through

$$\begin{aligned} I_{Pol} &\equiv -\text{Tr} \int dX \wedge * dX \\ &= \int \sqrt{-g} g^{AB} \eta_{\mu\nu} \partial_A X^\mu \partial_B X^\nu dx^0 dx^1, \end{aligned} \quad (1)$$

where capital roman indices run from 0 to 1, $d = dx^A \partial_A$ is the exterior derivative on M , $*$ is the Hodge $*$ -operator, and g_{AB} is the worldsheet metric. In this case X^μ describes a string.

The kinetic part of the Rasetti-Regge action is also constructed from X , but now

$$\begin{aligned} I_{RR} &\equiv -\frac{1}{3} \text{Tr} \int X dX \wedge dX = -\frac{1}{3} \text{Tr} \int \mathbf{X} d\mathbf{X} \wedge d\mathbf{X} \\ &= -\frac{1}{3} \int f_{abc} X^a \partial_0 X^b \partial_1 X^c dx^0 dx^1. \end{aligned} \quad (2)$$

X^μ in this case describes a vortex line. Note, however, that g_{AB} does not explicitly appear; I_{RR} is a topological invariant and describes a Schwarz-type TQFT similar to Chern-Simons theory. (This corresponds to an antisymmetric-field lagrangian in background-field string theory with $B_{ab} \sim \epsilon_{abc} X^c$ in three dimensions.) Note also that I_{RR} is translationally invariant; the lagrangian changes by a total derivative, $X dX \wedge dX \rightarrow X dX \wedge dX + \Xi d(X dX)$

under the uniform translation $X \rightarrow X + \Xi$. Indeed, I_{RR} is the *only* translationally invariant topological action that can be constructed directly from X^μ . In the special case of $\mathfrak{g} = so(3)$, I_{RR} is proportional to the lagrangian in [15], but without the coupling due to self-interaction.

Notice that I_{RR} does not depend on X^0 , the time component of X^μ . This is expected: topological lagrangians describe systems with no dynamical degrees of freedom. We will thus work solely with \mathbf{X} from this point on. This \mathbf{X} is a section of a the vector bundle \mathbb{R}^m over M , and is at the same time an element of \mathfrak{g} , a *vector* on \mathbb{R}^m , and a 0-form (and thus a function) on M . Seen thusly, the structure constants f_{abc} form a rank-3, totally antisymmetric tensor on \mathbb{R}^m . The 1-form $\mathbf{F} = \mathbf{F}_A dx^A = F_A^a dx^A \mathbf{T}_a$ is then a *vector valued* or, equivalently, a Lie algebra valued 1-form on M , meaning that each of its two *components* \mathbf{F}_A are both vectors in \mathbb{R}^m and members of \mathfrak{g} .

The equations of motion, $d\mathbf{X} \wedge d\mathbf{X} = 0$, from eq. (2) can be integrated once to give

$$\mathbf{P} \equiv [\mathbf{X}, d\mathbf{X}], \quad (3)$$

where \mathbf{P} is a *closed* Lie algebra valued 1-form on the worldsheet: $d\mathbf{P} = 0$. The two components of \mathbf{P} are $\mathbf{P}_0 \equiv P_0^c \mathbf{T}_c = f_{ab}{}^c X^a \partial_0 X^b \mathbf{T}_c$ and $\mathbf{P}_1 \equiv P_1^c \mathbf{T}_c = f_{ab}{}^c X^a \partial_1 X^b \mathbf{T}_c$. \mathbf{P} is related to the canonical momentum for \mathbf{X} through the dual form $\mathbf{\Pi} \equiv * \mathbf{P}$:

$$\Pi_a^A = \frac{1}{\sqrt{-g}} \frac{\delta I_{RR}}{\delta \partial_A X^a}, \quad (4)$$

where $\mathbf{\Pi} = \Pi_a^A dx^A \mathbf{T}_a$. The components of \mathbf{P} then determine the momentum of the vortex, and we call \mathbf{P} the *momentum 1-form*.

This choice for \mathbf{P} is only unique up to a total derivative. Although we could have just as well chosen $\mathbf{P}' = 2\mathbf{X}d\mathbf{X}$, $\mathbf{P} - \mathbf{P}' = d\mathbf{X}^2$, and the two choices differ by an exact form. Indeed, under uniform translations, $\mathbf{X} \rightarrow \mathbf{X} + \mathbf{K}$, $\mathbf{P} \rightarrow \mathbf{P} + d[\mathbf{K}, \mathbf{X}]$, and \mathbf{P} changes by an *exact* 1-form. Conversely, suppose we have $\mathbf{P}_1 \equiv [\mathbf{X}_1, d\mathbf{X}_1]$ and $\mathbf{P}_2 \equiv [\mathbf{X}_2, d\mathbf{X}_2]$ that differ by a close form $d\mathbf{F}$. Then $d\mathbf{F} = d[\mathbf{X}_2 - \mathbf{X}_1, \mathbf{X}_2 + \mathbf{X}_1]/2 - [\mathbf{X}_2 + \mathbf{X}_1, d(\mathbf{X}_2 - \mathbf{X}_1)]$ so that either $\mathbf{X}_2 - \mathbf{X}_1 = \mathbf{K}$, or $\mathbf{X}_2 + \mathbf{X}_1 = \mathbf{K}$, where \mathbf{K} is a constant. Thus, \mathbf{X}_2 is related to \mathbf{X}_1 by either a uniform translation or a reflection plus a translation. Therefore, what is physically relevant are the *equivalence classes* of \mathbf{P} , where $\mathbf{P}_1 \sim \mathbf{P}_2$ if they differ by an exact form, and not any one specific choice of \mathbf{P} . Consequently, $\mathbf{P} \in H^1(M, \mathbb{R}^m)$, the first cohomology class of M , and we are interested in \mathbf{P} that are closed but *not* exact.

The cohomology classes for 2-d surfaces are well known [35]. In particular, $H^1(M, \mathbb{R}^m) = 0$ if M is *not* a closed surface. This result has definite implications for the dynamics of

vortex lines: The dynamics of an open vortex line, which sweeps-out a 2-d open sheet in \mathbb{R}^m , differ dramatically from that of a closed vortex line (vortex ring), which sweeps out a closed surface.

For the open vortex line, $H^1(M, \mathbb{R}^m) = 0$ and we can always make a translation to a frame in which the momentum vanishes, $\mathbf{P} = 0$ so that $0 = [\mathbf{X}, d\mathbf{X}]$. The solution for \mathbf{X} in this case is particularly simple. For $\mathfrak{g} = so(3), su(2), sp(2)$, $\mathbf{X} = a(x^0, x^1)\mathbf{H}$, where a is an arbitrary function and \mathbf{H} is a constant vector. The vortex line is constrained to move along one direction: \mathbf{H} . For other Lie algebras, $\mathbf{X} \in \mathfrak{c}$, the Cartan subalgebra for \mathfrak{g} , so that $\mathbf{X} = X^i \mathbf{H}^i$ where $\{\mathbf{H}^i\}$ form the bases for \mathfrak{c} [34], and \mathbf{X} propagates within a linear subspace of \mathbb{R}^m .

For the vortex ring, on the other hand, $H^1(M, \mathbb{R}^m) = \mathbb{Z}^m$, the integers, and \mathbf{P} need not vanish. The dynamics of vortex rings are thus much more interesting, and we shall focus on them for the rest of the paper. We begin with $\mathfrak{g} = so(3), su(2)$ or $sp(2)$. The vortex is propagating in \mathbb{R}^3 and it's dynamics are especially constrained.

VORTEX RINGS IN \mathbb{R}^3

When $\mathfrak{g} = so(3)$, $f_{abc} = \epsilon_{abc}$, and we are dealing with a vortex line propagating in \mathbb{R}^3 . Although we can revert to the usual vector notation in this case, doing so will add notational complexity. Instead, we introduce a slight abuse of notation and write the cross product of two vectors $\mathbf{V}, \mathbf{U} \in \mathbb{R}^3$ as $\mathbf{V} \times \mathbf{U} \equiv [\mathbf{V}, \mathbf{U}]$.

It is straightforward to show that $[\mathbf{P}_0, \mathbf{P}_1] = 0$; the two vectors are proportional to one another. Consequently, we can write $\mathbf{P} = \hat{\mathbf{b}}p$ where p is a scalar 1-form on M and $\hat{\mathbf{b}} \in \mathbb{R}^3$. (The hat denotes a unit vector: $|\hat{\mathbf{b}}|^2 = \langle \hat{\mathbf{b}}, \hat{\mathbf{b}} \rangle = 1$.) Because $d\mathbf{P} = 0$, $d\hat{\mathbf{b}} \wedge p + \hat{\mathbf{b}}dp = 0$; each term must vanish separately. Consequently, $dp = 0$, and p is a closed 1-form. For the other term, $d\hat{\mathbf{b}} \wedge p = 0$, and from Cartan's lemma [36], $d\hat{\mathbf{b}}$ must be proportional to p : $d\hat{\mathbf{b}} = -\tau\hat{\mathbf{n}}p$, where τ is an arbitrary function and $\hat{\mathbf{n}}$ is a unit vector in \mathbb{R}^3 orthogonal to $\hat{\mathbf{b}}$. Doing this trick once again and noting that $dd\hat{\mathbf{b}} = 0$, $d\hat{\mathbf{n}} = (-\kappa\hat{\mathbf{t}} + \tau\hat{\mathbf{b}})p$ where κ is another arbitrary function on M and $\hat{\mathbf{t}} = \hat{\mathbf{n}} \times \hat{\mathbf{b}}$. Once again $\langle \hat{\mathbf{n}}, \hat{\mathbf{t}} \rangle = 0$. It is then straightforward to show that $d\hat{\mathbf{t}} = \kappa\hat{\mathbf{n}}$, and no more terms need to be introduced.

To complete the solution for \mathbf{X} , we note that $\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}$ form a moving orthogonal coordinate system on \mathbb{R}^3 . Taking $\mathbf{X} = |\mathbf{X}|(\alpha\hat{\mathbf{t}} + \beta\hat{\mathbf{n}} + \gamma\hat{\mathbf{b}})$ for constants α, β, γ , we require that this \mathbf{X}

solves eq. (3). Then $\alpha = 1$ and \mathbf{X} lies along $\hat{\mathbf{t}}$, while $|\mathbf{X}|^2 = 1/\kappa$. Solution of equations of motion therefore reduces to finding solutions for $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$, and $\hat{\mathbf{b}}$ for given κ , τ and p .

From $\langle \hat{\mathbf{b}}, \hat{\mathbf{n}} \rangle = 0$ and $d\hat{\mathbf{t}} = \kappa \hat{\mathbf{n}}$, we see that $d\tau$ and $d\kappa$ are both proportional to the 1-form p ; the functions κ , τ that determine $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$, $\hat{\mathbf{b}}$ all depend upon p . Consequently, there is a function $s(x^0, x^1)$ such that *locally* $ds = p$ and

$$\begin{aligned}\hat{\mathbf{t}}' &= \kappa(s)\hat{\mathbf{n}}, \\ \hat{\mathbf{n}}' &= -\kappa(s)\hat{\mathbf{t}} + \tau(s)\hat{\mathbf{b}}, \\ \hat{\mathbf{b}}' &= -\tau(s)\hat{\mathbf{n}},\end{aligned}\tag{5}$$

where the prime denotes derivative wrt to s . These are the Serret-Frenet equations [36] for a curve $\mathbf{c} : [a, b] \rightarrow \mathbb{R}^3$ parameterized by its arclength s . $\kappa = 1/|\mathbf{X}|^2$ is the local curvature of \mathbf{c} and is positive definite, as required, while τ is the local torsion. Because \mathbf{P} is a closed 1-form that is *not* exact, \mathbf{c} is a *closed* loop in \mathbb{R}^3 [36].

The existence of solutions to the Serret-Frenet equations is guaranteed [36]. It is nevertheless instructive to look further into their explicit form for two special cases. Let $\kappa = \varpi \cos u$, $\tau = \varpi \sin u$ where $u = u(s)$ and $-\pi/2 \leq u \leq \pi/2$ because $\kappa \geq 0$. Working with the coordinates $dt = \varpi ds$, eqs. (5) can be combined into $\ddot{\hat{\mathbf{n}}} = -\hat{\mathbf{n}} + \dot{u} \hat{\mathbf{n}} \mathbf{x} \dot{\hat{\mathbf{n}}}$, where the dot denotes derivative wrt t . This is similar to the equation of motion for a particle constrained to move on a sphere in the presence of a electric and magnetic dipole (a dyon) at the center of it, but in this case the ratio of the magnetic to electric “charge” of the dyon is \dot{u} and can depend on time. Taking $\hat{\mathbf{l}} = \hat{\mathbf{n}} \mathbf{x} \dot{\hat{\mathbf{n}}}$, the torque $\dot{\hat{\mathbf{l}}} = -\dot{u} \dot{\hat{\mathbf{n}}}$ is opposite of the velocity of the particle $\dot{\hat{\mathbf{n}}}$ and has strength \dot{u} . Consequently the total volume $\hat{\mathbf{n}} \cdot \dot{\hat{\mathbf{n}}} \mathbf{x} \ddot{\hat{\mathbf{n}}}$ swept out by $\hat{\mathbf{n}}$ is just \dot{u} . If this volume is a constant, then taking $\omega = \sqrt{1 + \dot{u}^2}$,

$$\begin{aligned}\hat{\mathbf{t}} &= \left\{ \cos u \sin(\omega t) - \frac{\dot{u}}{\omega} \sin u \cos(\omega t) \right\} \mathbf{T}_1 - \\ &\quad \left\{ \cos u \cos(\omega t) + \frac{\dot{u}}{\omega} \sin u \sin(\omega t) \right\} \mathbf{T}_2 + \frac{\sin u}{\omega} \mathbf{T}_3 \\ \hat{\mathbf{b}} &= - \left\{ \sin u \sin(\omega t) + \frac{\dot{u}}{\omega} \cos u \cos(\omega t) \right\} \mathbf{T}_1 - \\ &\quad \left\{ \sin u \cos(\omega t) - \frac{\dot{u}}{\omega} \cos u \sin(\omega t) \right\} \mathbf{T}_2 + \frac{\cos u}{\omega} \mathbf{T}_3 \\ \hat{\mathbf{n}} &= \frac{1}{\omega} \{ \cos(\omega t) \mathbf{T}_1 + \sin(\omega t) \mathbf{T}_2 + \dot{u} \mathbf{T}_3 \}.\end{aligned}\tag{6}$$

Furthermore, if $\dot{u} = 0$, then $\hat{\mathbf{b}} = \mathbf{T}_3$ and \mathbf{c} is confined to the 1 – 2 plane. Periodicity of $\hat{\mathbf{t}}$

and $\hat{\mathbf{n}}$ for a closed curve \mathbf{c} gives $2\pi n = \int_a^b \varpi ds = \int_a^b \varpi p$. This is a well-known result [36] for closed curves and is the fundamental reason why p (and consequently \mathbf{P}) is a close but not exact 1-form. For general u , $0 = \ddot{\hat{\mathbf{n}}} - \ddot{u}\dot{\hat{\mathbf{n}}}/\dot{u} + (1 + \dot{u}^2)\dot{\hat{\mathbf{n}}} - \ddot{u}\hat{\mathbf{n}}/\dot{u}$ with the boundary conditions $|\hat{\mathbf{n}}| = 1$, $|\dot{\hat{\mathbf{n}}}| = 1$, and $\hat{\mathbf{n}}(0) = \mathbf{T}^1$.

VORTEX RINGS IN \mathbb{R}^m

To solve eq. (3) for general \mathbf{g} we follow an approach similar to that in the previous section and introduce a set of linearly independent vectors $\{\hat{\mathbf{t}}_r\} \in \mathbb{R}^m$ on M where $\hat{\mathbf{t}}_r = R_r^a \mathbf{T}_a$ such that $\langle \hat{\mathbf{t}}_r, \hat{\mathbf{t}}_s \rangle = \delta_{r,s}$. The set $\{\hat{\mathbf{t}}_r\}$ forms a moving frame on \mathbb{R}^m for points on M (letters in the second have of the alphabet denote coordinates in the moving frame). Then $R_{ra}R_{sa} = \delta_{rs}$ and $R \in so(m)$; similarly, $R_{ra}R_{rb} = \delta_{ab}$. In addition, $[\hat{\mathbf{t}}_r, \hat{\mathbf{t}}_s] = f_{rs}^t \hat{\mathbf{t}}_t$, but now $f_{rs}^t = R_r^a R_s^b R_c^t f_{ab}^c$ are the “structure constants” in the moving frame. Because they depend on R_r^a , in this frame f_{rs}^t need not be constant.

Since $\{\hat{\mathbf{t}}_r\}$ are orthonormal and span \mathbb{R}^m , $d\hat{\mathbf{t}}_r = -\kappa_{rs}\hat{\mathbf{t}}_s$, where $\kappa_{rs} = -\kappa_{sr}$ are 1-forms on M . Moreover, from $dd\hat{\mathbf{t}}_r = 0$,

$$0 = d\kappa_{rs} + \kappa_{rt} \wedge \kappa_{ts}. \quad (7)$$

These are the Maurer-Cartan equations [36] and κ_{rs} are the Maurer-Cartan 1-forms for $so(m)$. Indeed, let $S^{\tilde{a}}$ be the generators of $so(m)$, the symmetry group of \mathbb{R}^m , such that $[S^{\tilde{a}}, S^{\tilde{b}}] = k^{\tilde{a}\tilde{b}}_{\tilde{c}} S^{\tilde{c}}$, $\text{Tr} \{S^{\tilde{a}} S^{\tilde{b}}\} = -\delta^{\tilde{a}\tilde{b}}$, and \tilde{a} runs from 1 to $m(m-1)/2$. For a fixed \tilde{a} , $S^{\tilde{a}}$ are $m \times m$ antisymmetric matrices with elements $(S^{\tilde{a}})_{rs}$ (we are *not* working in the adjoint representation for $so(m)$). We then introduce the 1-forms $A = A^{\tilde{a}} S^{\tilde{a}}$ with values in the Lie algebra $so(m)$ such that $\kappa_{rs} \equiv (A^{\tilde{a}} S^{\tilde{a}})_{rs}$. Then $dA + A \wedge A = 0$; A can be seen as a non-abelian “vector potential” for the group $so(m)$. The field strength for A vanishes, however, and A is a “pure gauge” vector potential. As expected, A does not contain any physical degrees of freedom. Indeed, written in terms of matrices of $so(m)$, $R^t dR = -A$.

With this interpretation of the Maurier-Cartan equations it is straightforward to see that

$$\hat{\mathbf{t}}_r = \text{P} \left(\exp \int_0^s A \right)_r^a \mathbf{T}_a, \quad (8)$$

where P denote Wilson path ordering.

Solution to eq. (3) now follows straightforwardly. Given a set of κ_{rs} , we construct $\hat{\mathbf{t}}_r$ using eq. (8). We then choose $\mathbf{X} = |\mathbf{X}|\hat{\mathbf{t}}_1$ so that $\mathbf{P} = |\mathbf{X}|^2 f_{1rs} \kappa_{1r} \hat{\mathbf{t}}_s$. Because $d\mathbf{P} = 0$,

$$0 = \{d \log |\mathbf{X}|^2 f_{1r's'} - \kappa_{1t'} f_{tr's'}\} \wedge \kappa_{1r'}, \quad (9)$$

where $r', s', t' > 1$ and we have used $df_{rst} = -\kappa_{rn} f_{nst} - \kappa_{sn} f_{rnt} - \kappa_{tn} f_{rsn}$. In addition, the choice of κ_{rs} must satisfy the constraint $0 = f_{1r's'} \kappa_{1r'} \wedge \kappa_{1s'}$; $\{\kappa_{1r'}\}$ therefore can *not* linearly independent. One solution of this constraint equation is $\kappa_{1r'} = \kappa_{r'} \pi$, where κ_a are functions on M and π is a 1-form on M . This choice of $\kappa_{1r'}$ does not restrict $\kappa_{r's'}$ and $0 = d\kappa_{r's'} + \kappa_{r't'} \wedge \kappa_{t's'}$ still. Integration of eq. (9) then gives $|\mathbf{X}|^2 = \exp\{\int \alpha \pi\}$ for any function α on M , and we are done. \mathbf{X} is determined by the arbitrary function α , and Maurer-Cartan 1-forms $\kappa_{r'} \pi$, and $\kappa_{r's'}$.

Except for $so(3)$, $su(2)$, and $sp(2)$, this choice of $\kappa_{1r'}$ is not the most general one that satisfies the constraint equation. Indeed, with this choice, $\mathbf{P}_0 \propto \mathbf{P}_1$ and like the \mathbb{R}^3 case, the two components of \mathbf{P} are proportional to one another. It is expected that when the general solution to the constraint equation is used, this relationship between the components of \mathbf{P} will no longer hold.

CONCLUDING REMARKS

We have shown in this paper the deep connection between I_{RR} , on the one hand, and string theory and TQFT on the other. Indeed, the topological nature of the theory, and the fundamental role it plays in determining vortex dynamics, is manifest in our approach in analyzing the system. Moreover, with this approach generalization of vortex dynamics to higher dimensions becomes straightforward.

With the goal being to establish the links between TQFT, string theory, and the study of vortex lines, the approach we have taken in this paper has been purposefully formal. We have focused on establishing mathematical structures, and using these structures in understanding the general physical properties of vortices propagating in superfluids that is due solely to the kinetic part of the full Rasette-Regge lagrangian.

We have focused on only the kinetic part of the lagrangian for two reasons. First, the traditional interaction term between vortex lines found in [15] is extremely nonlinear. Some degree of perturbative analysis, based on the kinetic term, would most likely be needed.

To this end, a thorough understanding of the “free” kinetic term is needed. Second, the interaction term has a $1/r$ type of divergence singularity at the classical level, which has traditionally been regulated by introducing a finite vortex core. However, it is expected that the degree of divergence will be weakened in the full quantum field theoretic treatment of the system, and a complete treatment of this divergence will most fruitfully be delayed until then. The first step in the quantum field theoretic approach is to the quantization of the “free” (kinetic) part of the Rasette-Regge lagrangian I_{RR} .

How to treat the many-vortex system is still an open question. Once more than one vortex line is introduced, a whole host of questions come to the fore. One particular issue is the question of how interactions between them should be incorporated into the approach outlined here. One can certainly choose to use the classical interaction term found in [15]. Another approach could be to follow the approach of string theory where the interaction of strings are represented by the merging and breaking of strings (which for closed strings fundamentally changes the genus, and thus topology, of the surface it sweeps-out). Much of the techniques developed for string theory could then conceivably be applied to the analysis of interacting vortex lines. Which of these two approaches will be more fruitful is unclear, especially in light of the two points listed above.

The question of how to include interactions between vortices goes beyond a discussion of field-theoretic techniques and methodology, however. As we have mentioned in the introduction, a TQFT has no dynamics in the traditional sense; since the lagrangian does not depend on the metric, there is no notion of time. Will the inclusion of the interaction terms necessitate the introduction of the metric? While it is possible to use de Rham’s method of generalized forms [37] to rewrite and generalize the interaction term found in [15] in terms of differential forms (which will thus automatically be independent of the metric), it is unclear if such an approach is physically meaningful. Moreover, making sense of this interaction term will require the introduction of a high energy cut-off (the vortex core size), which may bring along its own particular set of problems.

* Electronic address: adspelio@uclink.berkeley.edu

† Present address: Department of Physics, University of California at Berkeley, Berkeley, CA

94720-7300.

- [1] R. J. Donnelly, *Quantized Vortices in Helium II* (Cambridge University Press, New York, 1991).
- [2] H. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973).
- [3] L. Onsager, Nuovo Cimento Suppl. **6**, 249 (1949).
- [4] R. P. Feynman, in *Progress in Low Temperature Physics*, edited by C. Gorter (Horth-Holland, Amsterdam, 1955), vol. 1, p. 17.
- [5] G. A. Williams, Phys. Rev. Lett. **59**, 1926 (1987).
- [6] F. Lund, A. Reisenegger, and C. Utreras, Phys. Rev. **B41**, 155 (1990).
- [7] M. Anderson, J. Ensher, M. Matthews, C. Wieman, and E. Cornell, Science **269**, 198 (1995).
- [8] W. Ketterle and N. van Druten, Phys. Rev. A **54**, 656 (1996).
- [9] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **83**, 2498 (1999).
- [10] K. W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, Phys. Rev. Lett. **84**, 806 (2000).
- [11] F. Chevy, K. W. Madison, and J. Dalibard, Phys. Rev. Lett. **85**, 2223 (2000).
- [12] B. P. Anderson, P. C. Haljan, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **85**, 2857 (2000).
- [13] D. S. Rokhsar, Phys. Rev. Lett. **79**, 2164 (1997).
- [14] J. García-Ripoll, G. Molina-Terriza, V. Pérez-García, and et al., Phys. Rev. Lett. **87**, 140403 (2001).
- [15] M. Rasetti and T. Regge, Physica **A80**, 217 (1975).
- [16] M. Rasetti and T. Regge, *Lecture Notes in Physics*, vol. 201 (Springer-Verlag, New York, 1984).
- [17] G. A. Goldin, R. Menikoff, and D. H. Sharp, Phys. Rev. Lett **58**, 2162 (1987).
- [18] V. Penna and M. Spera, J. Math. Phys **30**, 2778 (1989).
- [19] U. K. Albertin and H. L. Morrison, Physica **A159**, 188 (1989).
- [20] U. K. Albertin and H. L. Morrison, J. Math. Phys. **31**, 1535 (1990).
- [21] G. A. Goldin, R. Menikoff, and D. H. Sharp, Phys. Rev. Lett. **67**, 3499 (1991).
- [22] G. A. Goldin and U. Moschella, J. Phys. **A28**, L475 (1995).
- [23] V. Penna and M. Spera, Phys. Rev. **B62**, 14547 (2000).
- [24] A. M. Polyakov, Phys. Lett. **B103**, 207 (1981).

- [25] Y. Nambu, Lectures at the Copenhagen Summer Symposium (1970).
- [26] T. Goto, Prog. Theor. Phys. **46**, 1560 (1971).
- [27] F. Lund and T. Regge, Phys. Rev. **D14**, 1524 (1976).
- [28] E. Witten, Commun. Math. Phys. **117**, 353 (1988).
- [29] E. Witten, Commun. Math. Phys. **117**, 351 (1989).
- [30] A. S. Schwarz, Lett. Math. Phys. **2**, 247 (1978).
- [31] R. Ricca, Phys. Rev. **A43**, 4281 (1991).
- [32] C. G. Callan, D. Friedan, E. J. Martinec, and M. J. Perry, Nucl. Phys. p. 593 (1985).
- [33] A. Giveon, E. Rabinovici, and G. Veneziano, Nucl. Phys. **B322**, 167 (1989).
- [34] F. J. Murray and C. Schweigert, *Symmetries, Lie Algebras and Representations: A Graduate Course for Physicists* (Cambridge University Press, New York, 1997).
- [35] R. Bott and L. W. Tu, *Differential Forms in Algebraic Geometry* (Springer-Verlag, New York, 1982).
- [36] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, vol. 2 (Publish or Perish, Wilmington, DE, 1970).
- [37] G. de Rham, *Differentiable Manifolds: Forms, Currents, Harmonic Forms* (Springer-Verlag, New York, 1984).